Vibrations of Infinitely Long Cylindrical Shells Under Initial Stress†

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Summary

The general bending theory of shells under the influence of initial stress presented recently by Herrmann and Armenákas[1] is applied in this investigation to study the effect of initial uniform circumferential stress, uniform bending moment and uniform radial shear on the dynamic response of an infinitely long (motion is independent of the axial coordinate) elastic circular cylindrical shell.

Symbols

x, θ, z	= initial coordinates of a point
R	= radius of middle surface of the shell
h	= shell thickness
$\mathbf{K}_x, \mathbf{K}_\theta, \mathbf{K}_z$	= unit vectors tangent to the undeformed coordinate lines at a point
$\bar{\mathbf{K}}_x, \bar{\mathbf{K}}_\theta, \bar{\mathbf{K}}_z$	= unit vectors tangent to the deformed coordinate lines
u_x, u_θ, u_z	= displacement components in the $\mathbf{K}_x, \mathbf{K}_\theta, \mathbf{K}_z$, direction, respectively
u, v, w	= displacement components of the middle surface, in the $\mathbf{K}_x$, $\mathbf{K}_\theta$, and $\mathbf{K}_z$ directions, respectively
ψ_x, ψ_θ	= angles of rotation of a normal to the middle surface in the z-x and z-θ planes, respectively
ψ_z	= thickness stretch
E_x, E_θ	= unit elongation of a line element originally in the x and θ direction, respectively
N	= initial uniform circumferential shell-stress
M	= initial uniform moment about x-axis
Q	= initial uniform radial shell shear force
$\Delta F_\theta, \Delta q$	= circumferential and radial components, respectively, of the change due to deformation of the initial shell surface tractions, taken per unit undeformed middle surface area
m_θ	= shell moment due to the circumferential traction
Δm_θ	= change due to the deformation of m_θ
p	= uniform lateral pressure
ρ	= mass density
t	= time
ω	= frequency of vibration
Δ	= frequency factor equal to $R\omega\sqrt{\rho(1-\nu^2)/E}$
$E_p = Eh/(1-\nu^2)$, $D = Eh^3/12(1-\nu^2)$	= shell compressional and flexural modulus, respectively
E, G	= Young's and shear modulus, respectively
ν	= Poisson's ratio
K_c, K_h, K_r	= buckling coefficients for constant, directional, hydrostatic, and centrally directed pressure, respectively
$I = h^3/12$	= moment of inertia

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Introduction

In a recent paper[1] the authors presented several linear theories of motion of elastic cylindrical shells subjected to a general state of initial stress, wherein the general deformed configuration (referred to as the final state of stress) is attained from the unstressed and unstrained state by passing through an intermediate equilibrium state of initial stress. These theories are more inclusive than others available heretofore, in that they contain not only the influence of uniform initial membrane stresses, but also the effect of initial moments and transverse shear forces. Moreover they take into account the possibility that the initial stress may be a function of the space coordinator.

In obtaining the equations of motion in Ref. 1, the contribution of the displacements associated with the initial stresses was disregarded. It was assumed, moreover, that the dependence on the thickness coordinate of the displacement components u_x, u_θ, u_z is linear. As will be indicated subsequently, the conventional assumption that the radial component of the displacement is not a function of the thickness coordinate leads, in some instances, to erroneous results. The retention of at least the first order static effect of the thickness stretch ψ_z is necessary.*

In the past, various versions of linear theories of motion have been employed in studying the effects, on the frequency of the radial mode, of initial, uniform, axial, or circumferential stress. For instance, on the basis of the Marguerre theory of shallow shells,[2] Reissner,[3] by omitting the effect of transverse and longitudinal inertia, obtained a simple expression for the frequency of the radial mode of a shell under initial, uniform, axial, and circumferential stress. In the case

* The notation used here is identical to that employed in Ref. 1 where the longitudinal, the circumferential, and the radial components of the displacement were approximated by

$$\bar{u}_x = u(x,\theta,t) + z\psi_x(x,\theta,t)$$

$$\bar{u}_\theta = v(x,\theta,t) + z\psi_\theta(x,\theta,t)$$

$$\bar{u}_z = w(x,\theta,t) + z\psi_z(x,\theta,t)$$

of cylindrical shells, it can be shown that the Marguerre equations may be obtained from the bending theory presented in Ref. 1 by disregarding $1/n^2$, $h^2/12R^2$, and $h^2h^2/12R^2$ as compared to unity and, therefore, Reissner's results cannot be used in establishing the frequencies of modes having a small number n of circumferential waves.

In another investigation, by using a shallow shell membrane theory but retaining the effect of longitudinal and transverse inertia, Reissner[4] established the frequency equation for shells under all around pressure (axial stress equals one-half circumferential) and compared the results with those of his aforementioned investigation, and with an approximate formula proposed by Serbin.[5] It is apparent, however, that the conclusions deduced from this comparison are again valid only for modes having a large number of circumferential waves.

A set of equations derived by Timoshenko[7] were employed by Fung, Sechler, and Kaplan[6] in establishing the frequency of vibrations for a shell under all around pressure. Due to geometric complexities, however, in these equations certain terms whose relative order of magnitude is $1/n^2$ are incorrect; for modes having a small number of circumferential waves, therefore, these equations cannot be relied upon to yield more accurate results than those obtained from the Marguerre theory of shallow shells. Moreover, in deriving the Timoshenko equations the stress-strain relations of Love's first approximation are used, resulting in nonsymmetric displacement equations of motion and thus precluding the existence of a strain energy potential.

In this investigation, the bending theory presented in Ref. 1 will be applied to study the effect of initial uniform circumferential shell stress, initial uniform bending moment, and initial radial shear, on the dynamic response of an infinitely long elastic circular cylindrical shell.

The uniform initial circumferential stress may be induced by uniform lateral pressure p. This pressure is assumed to be: (a) Constant Directional—the direction and the magnitude per unit original area will remain unchanged during deformation, (b) Hydrostatic—the direction and magnitude per unit original area will alter due to deformation, (c) Centrally Directed——during deformation, the magnitude per unit original area will remain constant, and the direction will remain toward the center of the shell. In all three cases, the applied load system is conservative (see Ref. 8). Although the character of the internal stresses induced by these slightly different pressures is identical, their effect on the frequencies of vibration may vary considerably. In the limiting case of buckling, as demonstrated by Stevens[10] and Boresi[9] and discussed by Bodner,[8] the critical load is affected significantly by the magnitude and the direction which the external pressure assumes in the process of buckling.

The uniform initial bending moment may be induced in a shell by bending a flat plate or a curved plate of different curvature, while a uniform initial radial shear may be induced by circumferential surface shear tractions, uniformly distributed over the internal and the external surface of the shell.

Equations of Motion

A cylindrical shell of infinite length, constant thickness and mean radius R is referred to a system of modified cylindrical coordinates x, θ, z; x is measured along the axis of the shell, θ along the circumference, and z in the direction perpendicular to the middle surface.

The equations of motion for this shell will be reduced from the three displacement equations, Eqs. (38), given in Ref. 1 by setting u equal to zero and by assuming that the displacement components are independent of the axial coordinate. These assumptions reduce the problem to a case of plane strain. The results of this analysis may be applied to a ring by setting Poisson's ratio equal to zero in the expressions for the compressional and flexural moduli.

The equations of motion are

$$\left.\begin{aligned}
&\left[\left(E_p + N + \frac{M}{R}\right)\frac{1}{R^2}\frac{\partial^2}{\partial\theta^2} - \frac{N + m_z}{R^2} - \rho h\frac{\partial^2}{\partial t^2}\right]v + \left[(E_p + 2N + m_z)\frac{1}{R^2}\frac{\partial}{\partial\theta} - \frac{M}{R^3}\frac{\partial^3}{\partial\theta^3} - \frac{Q}{R^2}\frac{\partial^2}{\partial\theta^2} - \frac{Q}{R^2}\right]w + \Delta F_\theta + \frac{\Delta m_\theta}{R} = 0 \\
&\left[(E_p + 2N + m_z)\frac{1}{R^2}\frac{\partial}{\partial\theta} - \frac{M}{R^3}\frac{\partial^3}{\partial\theta^3} + \frac{Q}{R^2} + \frac{Q}{R^2}\frac{\partial^2}{\partial\theta^2}\right]v + \left[\frac{E_p}{R^2} + \frac{D}{R^4} + \frac{N}{R^2} - \frac{M}{R^3} + \left(\frac{2D}{R^4} - \frac{2M}{R^3} - \frac{N}{R^2} - \frac{m_z}{R^2}\right)\frac{\partial^2}{\partial\theta^2} + \frac{D}{R^4}\frac{\partial^4}{\partial\theta^4} + \rho h\frac{\partial^2}{\partial t^2}\right]w - \frac{1}{R}\frac{\partial}{\partial\theta}(\Delta m_\theta) - \Delta q = 0
\end{aligned}\right\} \quad (1)$$

N is the initial uniform circumferential shell-stress; M is the initial uniform moment about the x-axis; Q is the initial uniform radial shell shear; ΔF_θ and Δq are the circumferential and radial components, respectively, of the change due to deformation of the initial shell surface tractions, taken per unit undeformed middle surface area; m_θ is the shell moment due to the initial circumferential surface traction; Δm_θ is the change induced by the deformation of the initial shell moment m_θ; m_z is equal to the sum of the products of the radial component of the initial surface traction and the z coordinate evaluated at the two surfaces of the shell.

Uniform Lateral Pressure

For a shell under the influence of uniform lateral

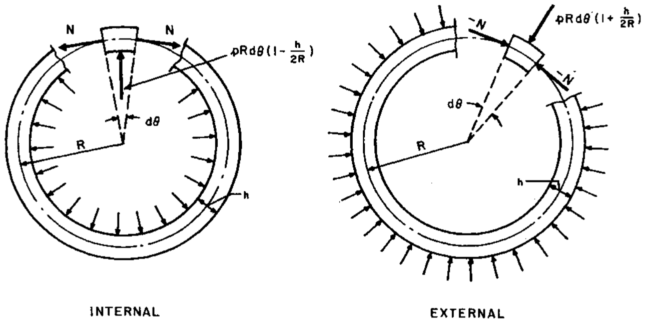

FIG. 1. Cross-section of shell under later pressure.

pressure it can be shown, referring to Fig. 1, that for initial equilibrium

$$N = \pm pR[1 \mp (h/2R)] \qquad (2)$$

The upper sign applies to internal pressure while the lower sign applies to external pressure. Furthermore, by definition, it is apparent that

$$m_z = -p(h/2)[1 \mp (h/2R)] \qquad (3)$$

For a shell under constant directional pressure, it is evident that

$$\Delta q = \Delta F_\theta = \Delta m_\theta = 0 \qquad (4)$$

In the case of hydrostatic pressure, referring to Fig. 2, it may be shown that

$$\left.\begin{aligned} \Delta q &= p\left(1 \mp \frac{h}{2R}\right)(1 + E_\theta) - p\left(1 \mp \frac{h}{2R}\right) \approx \frac{N}{R^2}\left[\frac{\partial v}{\partial \theta} + w\left(1 \pm \frac{h}{2R}\right) \pm \frac{h}{2R}\frac{\partial^2 w}{\partial \theta^2}\right] \\ \Delta F_\theta &= p\left(1 \mp \frac{h}{2R}\right)(1 + E_\theta)\sin(\mathbf{K}_z\tilde{\mathbf{K}}_z) \approx \frac{N}{R^2}\left(v - \frac{\partial w}{\partial \theta}\right) \\ \Delta m_\theta &= \mp\frac{Nh}{2R^2}\left(v - \frac{\partial w}{\partial \theta}\right) \end{aligned}\right\} \qquad (5)$$

In the case of centrally directed pressure, it may be shown that

$$\Delta q = 0, \quad \Delta F_\theta = Nv/R^2, \quad \Delta m_\theta = \mp Nhv/2R \qquad (6)$$

Suitable solutions of the following form are assumed

$$\left.\begin{aligned} v &= V\cos(n\theta)e^{i\omega t} \\ w &= W\sin(n\theta)e^{i\omega t} \end{aligned}\right\} \qquad (7)$$

Introduction of the above relations into the equations of motion, Eqs. (1), as modified in accordance with Eq. (4) for constant directional, Eqs. (5) for hydrostatic and Eqs. (6) for centrally directed pressure, results in two homogeneous algebraic equations for the amplitude factors V and W. For a nontrivial solution of these equations, the determinant of their coefficients is set equal to zero yielding the following frequency equations.

(a) Constant directional pressure

$$\omega^4 - \frac{\omega^2}{\rho h R^2}\left[E_p(1 + n^2) + \frac{D}{R^2}(n^2 - 1)^2 + N(1 + n^2)\left(2 \mp \frac{h}{2R}\right)\right] + \frac{N}{\rho^2 h^2 R^4}\left[E_p(n^2 - 1)^2 \times \left(1 \mp \frac{h}{2R}\right) + \frac{D}{R^2}(n^2 - 1)^2\left(n^2 + 1 \mp \frac{h}{2R}\right) + N(n^2 - 1)^2\left(1 \mp \frac{h}{2R}\right)\right] + \frac{E_p D}{\rho^2 h^2 R^6} n^2 \times (n^2 - 1)^2 = 0 \qquad (8)$$

(b) Hydrostatic pressure

$$\omega^4 - \frac{\omega^2}{\rho h R^2}\left[E_p(1 + n^2) + \frac{D}{R^2}(n^2 - 1)^2 + N\left[2n^2 \mp \frac{h}{2R}(1 - n^2)\right]\right] + \frac{N}{\rho^2 h^2 R^4} \times \left[E_p(n^2 - 1)n^2\left(1 \pm \frac{h}{2R}\right) + \frac{D}{R^2}n^2(n^2 - 1)^2 + Nn^2(n^2 - 1)\left(1 \pm \frac{h}{2R}\right)\right] + \frac{E_p D}{\rho^2 h^2 R^6} n^2 \times (n^2 - 1)^2 = 0 \qquad (9)$$

(c) Centrally directed pressure

$$\omega^4 - \frac{\omega^2}{\rho h R^2}\left[E_p(n^2 + 1) + \frac{D}{R^2}(n^2 - 1)^2 + N\left(1 + 2n^2 \mp \frac{n^2 h}{2R}\right)\right] + \frac{N}{\rho^2 h^2 R^4}\left[(E_p + N) \times \left[n^2(n^2 - 2) \pm n^2(1 - n^2)\frac{h}{2R}\right] + \frac{D}{R^2}n^2 \times (n^2 - 1)^2\right] + \frac{E_p D}{\rho^2 h^2 R^6} n^2(n^2 - 1)^2 = 0 \qquad (10)$$

The higher frequency (predominantly v motion) for the case of constant directional pressure, disregarding

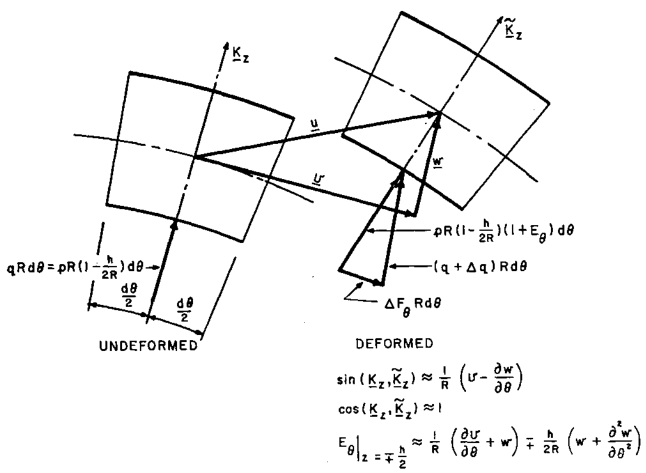

FIG. 2. Element of shell under hydrostatic pressure.

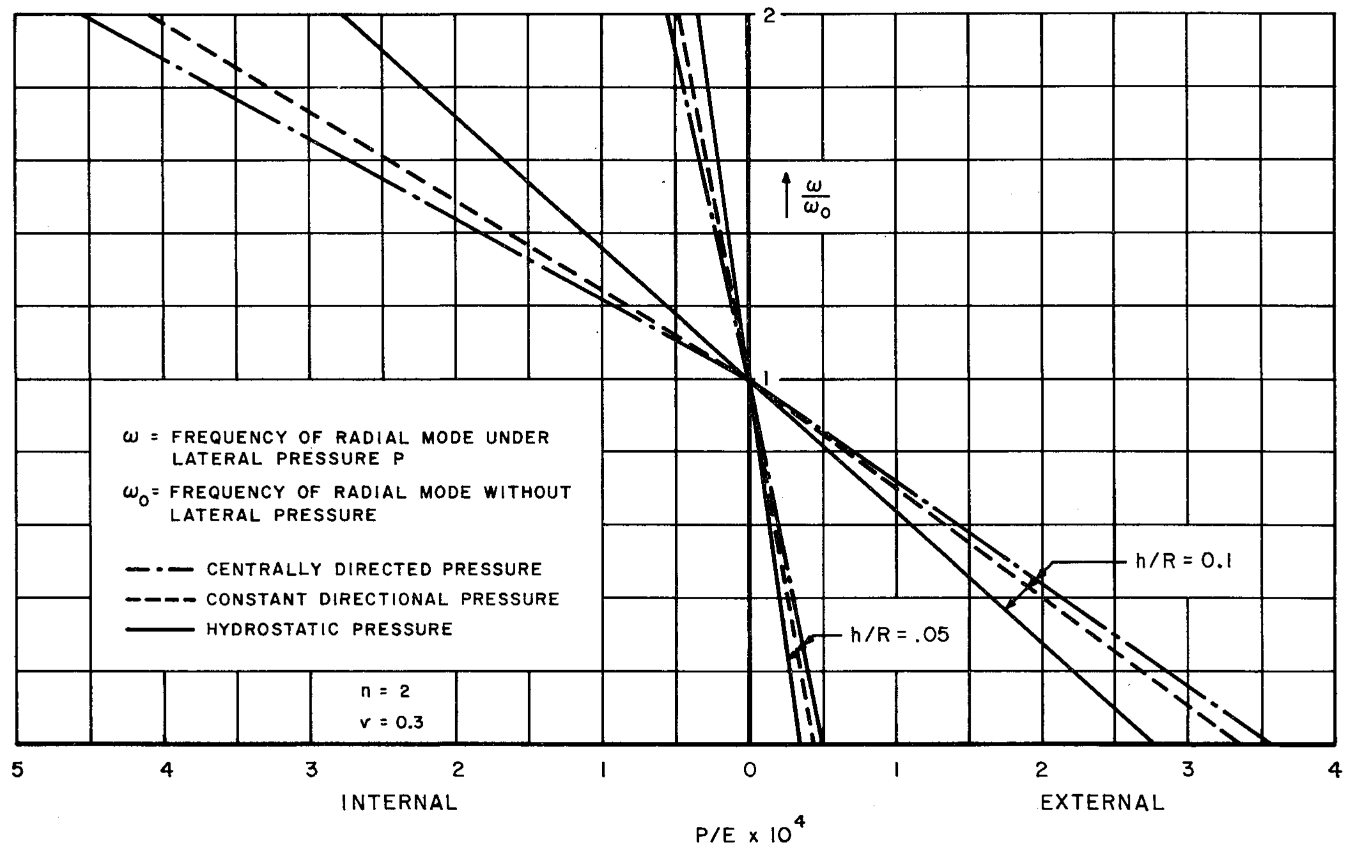

FIG. 3. Frequency of radial mode vs. lateral pressure.

terms of the order $(N/E_p)^2$ and $(h/R)^2$ as compared to unity, is

$$\omega^2 = \frac{E_p}{\rho h R^2}\left[(1+n^2) + \frac{h^2}{12R^2}(n^2-1)^2 + \frac{N}{E_p}(1+n^2)\left(2 \mp \frac{h}{2R}\right)\right] \quad (11)$$

It is apparent that this frequency increases with initial internal pressure and decreases with initial external pressure; the effect, generally, is very small inasmuch as for most engineering materials in the elastic range, N would be of the order of magnitude of $10^{-3}Eh$ or smaller. The influence on this frequency of hydrostatic and centrally directed pressure is similar to that of constant directional pressure.

The lower frequency, that is the frequency of the predominantly radial mode is, for the case of constant directional pressure

$$\omega^2 = \frac{Dn^2(n^2-1)^2}{\rho h R^4} \times \left[\frac{1 + \dfrac{NR^2}{Dn^2}\left(1 \mp \dfrac{h}{2R} + \dfrac{h^2}{12R^2}n^2\right)}{1 + n^2 + \dfrac{h^2}{12R^2}n^4}\right] \quad (12)$$

for the case of hydrostatic pressure

$$\omega^2 = \frac{Dn^2(n^2-1)^2}{\rho h R^4} \times \left[\frac{1 + [NR^2/D(n^2-1)]\left(1 \pm \dfrac{h}{2R} + \dfrac{h^2n^2}{12R^2}\right)}{1 + n^2 + n^4\dfrac{h^2}{12R^2}}\right] \quad (13)$$

for the case of centrally directed pressure

$$\omega^2 = \frac{Dn^2(n^2-1)^2}{\rho h R^4} \times \left[\frac{1 + \dfrac{NR^2}{D(n^2-1)^2}\left[n^2 - 2 \mp \dfrac{h}{2R}(n^2-1) + \dfrac{h^2n^4}{12R^2}\right]}{1 + n^2 + n^4\dfrac{h^2}{12R^2}}\right] \quad (14)$$

In obtaining Eqs. (12), (13), and (14), terms of the order of magnitude $(N/E_p)^2$ and $(h/R)^2$ have been disregarded as compared to unity. It may be observed that the frequency of the radial mode increases with initial internal pressure and decreases with initial external pressure. The relative effect becomes very large for very small values of h/R, as illustrated in Fig. 3, where the ratio of the frequency with lateral pressure, to that without lateral pressure, is plotted as a function of p/E for two values of h/R. The slight change of the slope of the curves at the origin is due to the change of the pressure at that point from external to internal. It is apparent, that for $h/R = 0.1$ this effect is considerable in the case of centrally directed pressure, small in the case of constant directional pressure, and negligible in the case of hydrostatic pressure. From the above equations it is evident that the relative effect of lateral pressure is more pronounced for modes having small numbers of circumferential waves. In Fig. 4, the frequency factor $\Delta = \omega \sqrt{\rho R^2(1-\nu^2)/E}$ is plotted as a function of h/R for two values of n and for a conveniently chosen value of N/Eh. It is apparent that the hydrostatic pressure has a greater influence on the frequency of the predominantly radial mode than the constant directional or centrally directed pressure. As indicated in Fig. 4, the difference of magnitude of the effect of these types of initial pressure on the frequency is more pronounced for small values of n; for $n = 6$ for instance, it becomes negligible.

In the case of thin shells, for modes with a large number of circumferential waves $1/n^2$ and $h^2n^2/12R^2$ may be disregarded as compared to unity and the expression for the frequencies, Eqs. (12), (13), and (14), reduces to

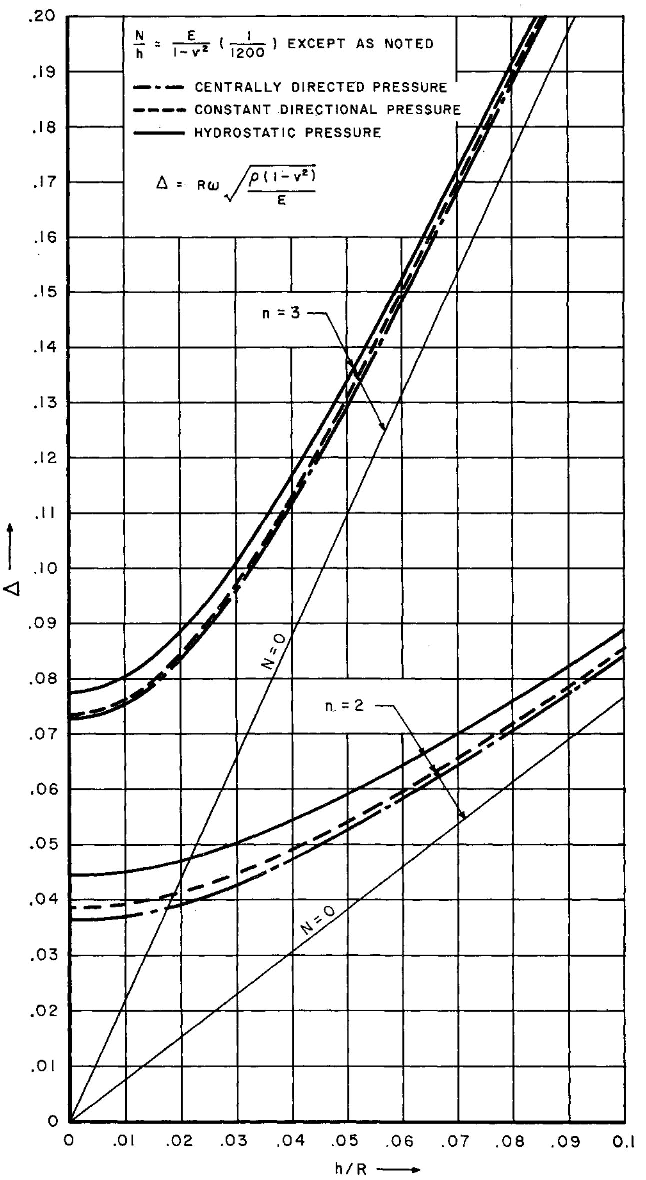

Fig. 4. Frequency factor Δ vs. h/R of the predominantly radial mode for an infinitely long cylindrical shell under internal pressure.

Table 1. Values of Buckling Coefficients

h/R	ν	K_c	K_h	K_i
0	0	4.00	3.00	4.50
	0.3	4.40	3.30	4.95
0.05	0	3.80	3.00	4.23
	0.3	4.18	3.30	4.65
0.10	0	3.61	3.00	3.85
	0.3	3.97	3.30	4.23

$$\omega^2 = Dn^4/h\rho R^4[1 + (NR^2/Dn^2)] \tag{15}$$

This result is identical to that obtained by Reissner[3] on the basis of the Marguerre shallow shell theory by disregarding the effects of axial and circumferential inertia. It may be concluded, therefore, that in the case of an infinitely long shell, the omission of circumferential inertia results in an error of the order of magnitude of $1/n^2$, which is justified for shallow shells and for circumferentially multiwaved modes. This result was anticipated on the basis of Reissner's conclusion.[11]

From the above equations, it can be seen that the frequency does not remain real for all values of the external pressure; consequently, the shell becomes unstable when the value of the pressure equals

$$p_{cr} = K(EI/R^3) \tag{16}$$

where

$$\left.\begin{aligned} K_c &= \frac{n^2}{(1-\nu^2)[1+(h/R)]} \\ K_h &= (n^2-1)/(1-\nu^2) \\ K_r &= \frac{(n^2-1)^2}{(1-\nu^2)\left[n^2 - 2 - (3-2n^2)\dfrac{h}{2R}\right]} \end{aligned}\right\} \tag{17}$$

In the above expressions $h^2n^2/12R^2$ is disregarded as compared to unity. The critical pressure is obtained for $n = 2$ since $n = 1$ represents a translation of the infinitely long shell as a rigid body. Values of K_c, K_h, and K_r for $\nu = 0$ and $\nu = 0.3$ for indicated values of h/R are given in Table 1. These results concur with those obtained by Boresi[9] for the case of hydrostatic and centrally directed pressure.

It is of interest to note, that in the case of constant directional pressure for axisymmetric vibrations ($\partial/\partial\theta = 0$), a circumferential mode is obtained which is nonexistent in the absence of initial circumferential tension. For this case, the equations of motion uncouple and the first reduces to

$$N[1 \mp (h/2R)]v + \rho hR^2(\partial^2 v/\partial t^2) = 0 \tag{18}$$

In this mode all points of the middle surface are displaced in the tangential direction by an amount equal to v and all crosssections rotate about their neutral

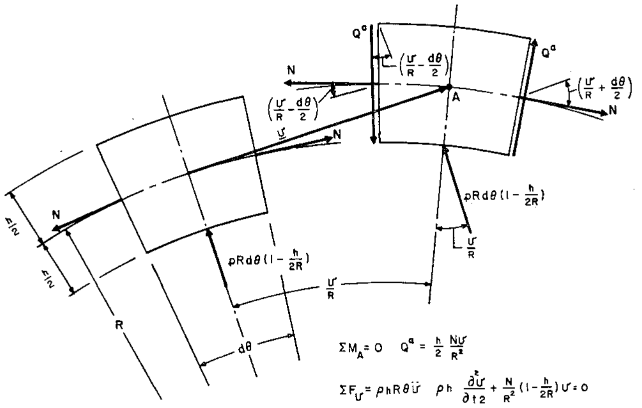

FIG. 5. Element of shell vibrating in pure circumferential mode.

axis by an angle approximately equal to v/R. The frequency of this mode is

$$\omega^2 = (N/\rho h R^2)[1 \mp (h/2R)] \qquad (19)$$

Eq. (18) may be readily obtained by considering the shell element shown in Fig. 5, and by setting the sum of the moments about any point equal to zero and the sum of the forces in the v direction equal to the v component of the change of linear momentum.

Uniform Moment and Hydrostatic Pressure

An initial uniform bending moment M may be induced in a shell for example during fabrication, if the shell is constructed from a flat plate or a shell of different curvature. The equations of motion for a shell under lateral hydrostatic pressure in addition to an initial constant moment M, may be readily obtained from Eqs. (1), (2), (3), and (5). Using solutions, Eq. (7), and equating to zero the determinant of the coefficients of the resulting linear algebraic equations in V and W, the following frequency equation is obtained

$$\omega^4 - \frac{\omega^2}{\rho h R^2}\left\{E_p(1+n^2) + \frac{D}{R^2}(n^2-1)^2 + N\left[2n^2 \mp \frac{h}{2R}(1-n^2)\right] + \frac{M}{R}(3n^2-1)\right\} + \frac{N}{\rho^2 h^2 R^4}\left\{E_p(n^2-1)n^2\left(1 \pm \frac{h}{2R}\right) + \frac{D}{R^2} \times (n^2-1)n^2 + N(n^2-1)n^2\left(1 \pm \frac{h}{2R}\right) + \frac{M}{R}\left[n^4 - 2n^3 + 2n^2 - 1 \pm \frac{hn^2}{2R}(n^2-1)\right]\right\} + \frac{M}{\rho^2 h^2 R^5}\left[\frac{D}{R^2}n^2(n^2-1)^2 - \frac{M}{R}n^2(n^2-1)^2\right] + \frac{E_p D n^2(n^2-1)^2}{\rho^2 h^2 R^6} = 0 \qquad (20)$$

By setting $N = 0$ in the above equation, it may be shown that the frequency of the predominantly circumferential mode increases due to initial positive moment by a very small amount. Disregarding $(M/E_p)^2$ and h^2/R^2 as compared to unity the frequency of the predominantly radial mode is

$$\omega^2 = \frac{Dn^2(n^2-1)^2}{\rho h R^4}\left[\frac{1 + (M/E_p R)}{1 + n^2 + n^4(h^2/12R^2)}\right] \qquad (21)$$

It may be seen that an initial positive moment, the one producing compression in the interior of the shell, increases this frequency; the effect, however, is generally very small, for in order that the stress not exceed the elastic limit, M must not exceed (for most engineering materials) the order of magnitude of $10^{-3}(Gh/2)$.

From Eq. (20) the critical lateral pressure at buckling is

$$P_{cr} = \frac{D(n^2-1)}{R^3}\left[1 + \frac{M}{D}\frac{h^2}{12R}\right] \qquad (22)$$

In obtaining the above equation terms of the order h^2/R^2, $(M/E_p)^2$, $(N/E_p)^2$, and $MN/E_p{}^2$ have been disregarded as compared to unity. If the shell were constructed from a flat plate, the initial moment due to bending of the plate will be $M = D/R$. For this case

$$p_{cr} = (D/R^3)(n^2-1)[1 + (h^2/12R^2)] \qquad (23)$$

It can be seen that the effect of this initial moment on the critical pressure is negligible. This result conflicts with the findings of Alfutov[12] wherein he has shown that the magnitude of the critical pressure at buckling of a ring made from a flat strip will increase by 25 percent due to the initial moment. A careful analysis reveals that in the strain energy expression in Ref. 12, the effect of the z component of strain has been disregarded. This effect contributes to the equations of equilibrium terms involving initial moments which cancel certain retained terms responsible for this erroneous result.

Uniform Radial Shear

The uniform initial radial shear may be induced by

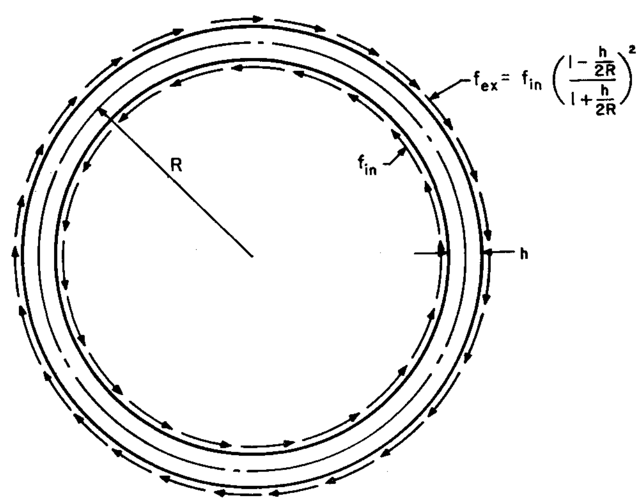

FIG. 6. Shell under uniform radial shear force.

circumferential surface shear tractions uniformly distributed over the internal and external surfaces of the shell which have relative magnitudes that satisfy the initial equilibrium conditions (see Fig. 6). Moreover, it is assumed that the magnitude of the resultant force acting on a given surface element does not change during deformation. The direction of the applied surface shear traction in the deformed position could be taken as either parallel to the undeformed center line or tangent to the elastic curve; within the accuracy of the present theory, the results will be identical.

For this case the equations of motion, Eqs. (1), reduce to

$$\left(E_p \frac{\partial^2}{\partial\theta^2} - \rho h R^2 \frac{\partial^2}{\partial t^2}\right) v + \left(E_p \frac{\partial}{\partial\theta} - Q \frac{\partial^2}{\partial\theta^2} - Q\right) w = 0$$

$$\left(E_p \frac{\partial}{\partial\theta} + Q \frac{\partial^2}{\partial\theta^2} + Q\right) v + \left[E_p + \frac{D}{R^2} \times \left(1 + 2\frac{\partial^2}{\partial\theta^2} + \frac{\partial^4}{\partial\theta^4}\right) + \rho h R^2 \frac{\partial^2}{\partial t^2}\right] W = 0 \quad (24)$$

For axi-symmetric vibrations, it may be shown that a predominantly circumferential mode does not exist and that the initial radial shear does not significantly effect the predominantly radial mode.

For the more general case of nonaxially symmetric motion, by disregarding the effect of circumferential inertia, the following relation may be obtained from Eqs. (24)

$$\left[\frac{\partial^6}{\partial\theta^6} + \left(2 + \frac{R^2Q^2}{E_pD}\right)\frac{\partial^4}{\partial\theta^4} + \left(1 + \frac{2R^2Q^2}{E_pD}\right)\frac{\partial^2}{\partial\theta^2} + \frac{Q^2R^2}{E_pD}\right] w = -\frac{\rho h R^4}{D}\frac{\partial^4 w}{\partial 6^2\,\partial t^2} \quad (25)$$

A solution is assumed of the form

$$w = W \sin(n\theta)e^{i\omega t} \quad (26)$$

which substituted in Eq. (25) yields the following expression for the frequency of the predominantly radial mode

$$\omega^2 = \frac{D(n^2-1)^2}{h\rho R^4}\left[1 - \frac{R^2Q^2}{E_pDn^2}\right] \quad (27)$$

It may be seen that the initial radial shear force decreases the frequency of this mode—an effect which may become very large for small values of h/R. Notice that for $Q = 0$ Eq. (27) approximates the frequency of the radial mode to the order of accuracy that $[1 + (1/n^2)]$ approximates unity, which again illustrates that the omission of the circumferential inertia results in an error of order of magnitude of $1/n^2$. From Eq. (27) it may be seen that the shell buckles in a fashion analogous to a shell under lateral pressure for

$$Q_{cr} = (n/R)\sqrt{E_pD} = (2/R)\sqrt{E_pD} \quad (28)$$

In order for a shell to buckle within the elastic region, the uniform radial shear may not exceed (for most common engineering materials) the order of magnitude of $10^{-3}Eh$; for this value of Q a shell buckles if its h/R ratio is about 0.0017. It is apparent, therefore, that buckling will occur within the elastic region for values of h/R which have some practical significance. It is rather difficult, however, to induce a state of uniform radial shear stress without impeding radial displacement.

This buckling phenomenon is analogous to that observed by the authors in the case of a plate or a beam under uniform transverse shear. (See Refs. 13 and 14.)

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